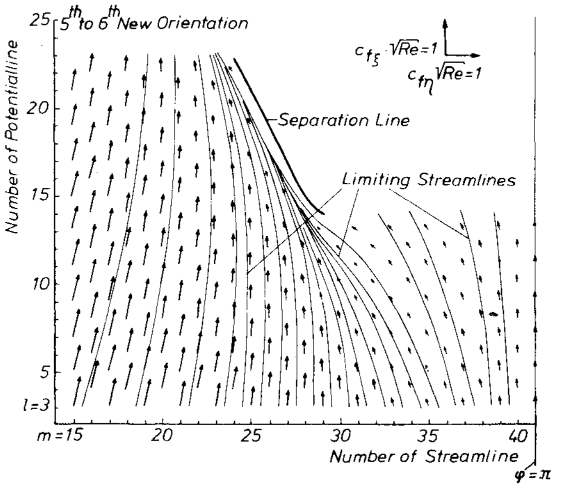

Fig. 4 Vectors of the local skin friction coefficient c_f and limiting streamlines. Beginning of separation between the 5th and 6th new orientation in Fig. 2, that is between $x/L = 0.251$ and $x/L = 0.385$ along the windward symmetry line.

$\alpha = 15°$. Both the profiles of the primary velocity, u, and of the secondary velocity, v, are plotted against the dimensionless distance ζ from the wall for five and three different streamlines, respectively, the numbers of which are given in Fig. 2. The area covered is close to the separation line of Fig. 2. In the region of maximum negative secondary flow (point $m = 30$ in Fig. 3) the u-profiles have a point of inflection corresponding to a minimum of wall shearing stress in this direction.

Separation

The numerical boundary-layer calculation can be extended over the body surface until a point is reached, where the stability condition Eq. (8) is violated. Figure 4 represents an example of the numerical calculation again for the ellipsoid of axis ratio $a/b = 4$ at an angle of attack of $\alpha = 15°$ (Fig. 2). In a small region which is marked in Fig. 2 the distribution of the vector of the local skin friction coefficient c_f is plotted as defined by

$$\tfrac{1}{2}c_f = \frac{\tau_o}{\rho U_\infty{}^2} = \frac{1}{(Re)^{1/2}}\left[\left(\frac{\partial u}{\partial \zeta}\right)^2 + \left(\frac{\partial v}{\partial \zeta}\right)^2\right]^{1/2}_{\zeta \to 0} \tag{9}$$

with τ_o as shearing stress at the wall.

The limiting streamlines are calculated by integration of the directions of the c_f-components. These wall streamlines run into one line forming an envelope. Near this envelope a maximum of reversed secondary flow in connection with a decrease of the primary velocity components occurs (see Fig. 3). The boundary-layer thickness reaches a steep maximum in this region. The computation can be extended downstream of the first numerical instability point always marching along equipotential lines until numerical instability occurs. The so determined instability line corresponds to the envelope of the limiting streamlines and is plotted in Fig. 2. This line is interpreted as the separation line of the free vortex layer type in the sense of Maskell[7] and Wang.[3] The separation line ends at the windward symmetry line of the potential flow, where reversed flow occurs in the boundary layer.

Table 1 Streamline station coordinates

m	x/L	ϕ[°]
1	0.360	0
13	0.322	71
25	0.277	122
30	0.264	141
41	0.254	180

Similar results have been obtained for numerous bodies of revolution with different axis ratios and angles of attack.[8] The present numerical method has also been tested for the simple cases of a sphere and a semi-infinite cylinder in cross flow. In both cases very satisfactory results with respect to the location of the separation point or separation line have been obtained with analytical methods.

References

[1] Blottner, F. G. and Ellis, M. A., "Finite-Difference Solution of the Incompressible Three-Dimensional Boundary Layer Equations for a Blunt Body," *Computers and Fluids*, Vol. 1, Pergamon, New York, 1973, pp. 133–158.

[2] Wang, K. C., "Three-Dimensional Boundary Layer near the Plane of Symmetry of a Spheroid at Incidence," *Journal of Fluid Mechanics*, Vol. 43, Pt. 1, Aug. 1970, pp. 187–209.

[3] Wang, K. C., "Separation Patterns of Boundary Layer over an Inclined Body of Revolution," *AIAA Journal*, Vol. 10, No. 8, Aug. 1972, pp. 1044–1050.

[4] Geissler, W., "Berechnung der Potentialströmung um rotationssymmetrische Rümpfe, Ringprofile und Triebwerkseinläufe," *Zeitschrift für Flugwissenschaften*, Bd. 20, 1972, pp. 457–462.

[5] Howarth, L., "The Boundary Layer in Three-Dimensional Flow, Pt. 2: The Flow near a Stagnation Point," *Philosophical Magazine*, Vol. 42, Ser. 7, No. 335, Dec. 1951, pp. 1433–1440.

[6] Hall, M. G., "A Numerical Method for Calculating Steady Three-Dimensional Laminar Boundary Layers," RAE TR 67145, Royal Aircraft Establishment, Farnborough, England, June 1967, pp. 1–23.

[7] Maskell, E. C., "Flow Separation in Three Dimensions," RAE Rept. Aero 2565, Royal Aircraft Establishment, Farnborough, England, 1955.

[8] Geissler, W., "Berechnung der Dreidimensionalen Laminaren Grenzschicht an Schrägangeströmten Rotationskörpern mit Ablösung," *Ingenieur Archiv*, to be published.

Penny-Shaped Crack in a Linear Viscoelastic Medium under Torsion

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I. Introduction

It is well known that to solve a viscoelastic boundary value problem that involves the boundary conditions of mixed type, the classical correspondence principle of obtaining the viscoelastic solution from the associated elastic solution is, in general, no longer applicable. Graham[1] has proposed a correspondence principle for problems with time-dependent boundary regions and has applied it[2] to solve the problems of a penny-shaped crack subjected to a tension normal to the plane of the crack. Ting[3] has developed a technique of solving problems with moving boundaries and has applied it to obtain the contact stresses between an axisymmetric rigid identer and a viscoelastic half space. Present authors[4] have applied this method to solve the problems of a penny-shaped crack in a viscoelastic medium under shear.

In the present Note we solve the related problem of a penny-shaped crack in a linear viscoelastic medium under torsion. Expressions for the stress distribution in the plane of the crack, displacement over the surface of the crack, and the stress intensity factor are given in the closed form. These quantities

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are then specialized for a Maxwell material and the effect of viscoelasticity, wherever possible, is pointed out.

II. Basic Equations and Formulation of the Problem

We write down the Laplace transforms of the relevant field equations and the constitutive equations appropriate to the quasistatic theory of linear viscoelasticity. For details of definitions and notations we refer to Refs. 1–4. These are:

$$2\bar{e}_{ij}(\mathbf{x}, s) = \bar{u}_{i,j}(\mathbf{x}, s)+\bar{u}_{j,i}(\mathbf{x}, s) \tag{1}$$

$$\bar{\sigma}_{ij,j}(\mathbf{x}, s) = 0 \qquad \bar{\sigma}_{ij}(\mathbf{x}, s) = \bar{\sigma}_{ji}(\mathbf{x}, s) \tag{2}$$

$$\bar{\sigma}_{ij}(\mathbf{x}, s) = s\bar{G}_1(s)\bar{e}_{ij}(\mathbf{x}, s)+\tfrac{1}{3}\delta_{ij}s(\bar{G}_2-\bar{G}_1)(s)\bar{e}_{kk}(\mathbf{x}, s) \tag{3}$$

Consider an infinite linear viscoelastic medium containing a penny shaped crack which is under torsion. We choose cylindrical polar coordinates (ρ, θ, z) such that the crack occupies the region $0 \leqq \rho \leqq a(t)$, $z = 0$ for all θ. The condition associated with the torsion then becomes

$$\sigma_{\theta z}(\rho, z, t) \sim S(\rho, t) \quad \text{as} \quad (\rho^2+z^2)\to\infty \tag{4}$$

We shall first solve the problem in a half space when the stress $\sigma_{\theta z}(\rho, 0, t) = -S(\rho, t)$ is acting on the crack surface, the entire crack surface is free from normal tractions and the displacement components exterior to the crack surface are all zero. In such stress distribution the displacement components u_ρ and u_z and hence their Laplace transform are identically zero. The equation of equilibrium for $\bar{u}_\theta$ and the boundary conditions now take the form

$$\partial^2\bar{u}_\theta/\partial\rho^2+(1/\rho)(\partial\bar{u}_\theta/\partial\rho)-\bar{u}_\theta/\rho^2+\partial\bar{u}_\theta/\partial z^2 = 0 \tag{5}$$

$$\sigma_{\theta z}(\rho, z, t) = 0 \quad \text{as} \quad (\rho^2+z^2)\to\infty$$

$$u_\theta(\rho, 0, t) = \begin{cases} 0 & \rho > a(t) \\ v(\rho, 0, t) & \rho \leqq a(t) \end{cases} \tag{6}$$

where $v(\rho, 0, t)$ is the unknown crack surface function. We also assume that $a(t)$ is a monotonic-increasing function of time.

On taking the Hankel transform of order one of Eq. (5) and solving the resulting equation we obtain

$$\hat{u}_\theta(\xi, z, s) = A\,e^{-\xi z}+B\,e^{\xi z} \tag{7}$$

where

$$\hat{u}_\theta(\xi, z, s) = H_1\{\bar{u}_\theta(\rho, z, s); \rho\to\xi\} = \int_0^\infty \rho\bar{u}_\theta(\rho, z, s)J_1(\xi\rho)\,d\rho \tag{8}$$

On employing the boundary conditions (6) in Eq. (7) we get

$$B = 0$$
$$\hat{u}_\theta(\xi, z, s) = \hat{v}(\xi, z, s)\,\bar{e}^{\xi z} \tag{9}$$

where $\hat{v}(\xi, z, s)$ denotes the Hankel transform of the Laplace transform of $v(\rho, z, t)$. If we employ similar notation for $\hat{\sigma}_{\theta z}$, it follows from Eqs. (1), (3), and (9) that

$$\hat{\sigma}_{\theta z}(\xi, 0, s) = -\frac{s\xi}{2}\bar{G}_1(s)\hat{v}(\xi, 0, s) \tag{10}$$

On taking the inverse Hankel and Laplace transforms, respectively, of Eq. (10), we obtain

$$\sigma_{\theta z}(\rho, 0, t) = -(1/2)\int_0^t G_1(t-\tau)\frac{\partial}{\partial\tau}\int_0^\infty \xi^2 J_1(\xi\rho)\,d\xi \times \int_0^{a(\tau)} \lambda v(\lambda, 0, \tau)J_1(\xi\lambda)\,d\lambda\,d\tau \tag{11}$$

Finally on employing the condition that $\sigma_{\theta z}(\rho, 0, t) = -S(\rho, t)$ when $\rho \leqq a(t)$, we get

$$2S(\rho, t) = \int_0^t G_1(t-\tau)\frac{\partial}{\partial\tau}\int_0^\infty \xi^2 J_1(\xi\rho)\,d\xi \times \int_0^{a(\tau)} \lambda v(\lambda, 0, \tau)J_1(\xi\lambda)\,d\lambda\,d\tau \qquad \rho \leqq a(t) \tag{12}$$

In Sec. III we shall give the solution of Eq. (12) for general loading $S(\rho, t)$ and then determine the various quantities of interest.

III. Solution of Integral Equation and Discussion

On changing the variable ρ to m in Eq. (12) and multiplying both sides of the equation by $[m^2/(\rho^2-m^2)^{1/2}]$ and integrating as follows, we get

$$2\int_0^\rho \frac{m^2 S(m, t)}{(\rho^2-m^2)^{1/2}}\,dm = \int_0^t G_1(t-\tau)\frac{\partial}{\partial\tau}\int_0^\infty \xi^2\,d\xi \times \int_0^\rho \frac{m^2 J_1(\xi m)}{(\rho^2-m^2)^{1/2}}\,dm \int_0^{a(\tau)} \lambda v(\lambda, 0, \tau)J_1(\xi\lambda)\,d\lambda\,d\tau \quad \rho \leqq a(t) \tag{13}$$

We point out that in writing Eq. (13) we have also switched the order of integration. On employing the following result from Watson[5]

$$\int_0^\rho \frac{m^2 J_1(\xi m)}{(\rho^2-m^2)^{1/2}}\,dm = \frac{\sin(\xi\rho)}{\xi^2}-\frac{\rho\cos(\xi\rho)}{\xi} \tag{14}$$

in Eq. (13), and again interchanging the order of integration yields

$$2\int_0^\rho \frac{m^2 S(m, t)}{(\rho^2-m^2)^{1/2}}\,dm = \int_0^t G_1(t-\tau)\frac{\partial}{\partial\tau}\left[1-\rho\frac{\partial}{\partial\rho}\right] \times \int_0^{a(\tau)} \lambda v(\lambda, 0, \tau)\,d\lambda \int_0^\infty J_1(\xi\lambda)\sin(\xi\rho)\,d\xi, \qquad \rho \leqq a(t) \tag{15}$$

On evaluating the last integral on the right-hand side of Eq. (15), again by using Watson,[5] and further simplifying we arrive at

$$2\int_0^\rho \frac{m^2 S(m, t)}{(\rho^2-m^2)^{1/2}}\,dm = -\int_0^t G_1(t-\tau)\frac{\partial}{\partial\tau} \times \left[\rho^2\frac{\partial}{\partial\rho}\int_0^{a(\tau)} \frac{v(\lambda, 0, \tau)}{(\lambda^2-\rho^2)^{1/2}}\,d\lambda\right]d\tau \qquad \rho \leqq a(t) \tag{16}$$

Now again, if we change the variable ρ to η and multiply both sides of Eq. (16) by $[(\eta^2-\rho^2)^{1/2}/\eta^2]$ and integrate as follows, we get

$$2\int_\rho^{a(t)} \frac{(\eta^2-\rho^2)^{1/2}}{\eta^2}\,d\eta \int_0^\eta \frac{m^2 S(m, t)}{(\eta^2-m^2)^{1/2}}\,dm = -\int_0^t G_1(t-\tau)\frac{\partial}{\partial\tau}\int_\rho^{a(t)} (\eta^2-\rho^2)^{1/2}\,d\eta \times \frac{\partial}{\partial\eta}\int_\eta^{a(\tau)} \frac{v(\lambda, 0, t)}{(\lambda^2-\rho^2)^{1/2}}\,d\lambda\,d\tau \qquad \rho \leqq a(t) \tag{17}$$

The right-hand side of Eq. (17), on integrating by parts and interchanging the order of integration, becomes equal to

$$\int_0^t G_1(t-\tau)\frac{\partial}{\partial\tau}\int_\rho^{a(t)} v(\lambda, 0, \tau)\,d\lambda \int_\rho^\lambda \frac{\eta\,d\eta}{(\eta^2-\rho^2)^{1/2}(\eta^2-\lambda^2)^{1/2}}\,d\tau = \frac{\pi}{2}\int_0^t G_1(t-\tau)\frac{\partial}{\partial\tau}\int_\rho^{a(t)} v(\lambda, 0, \tau)\,d\lambda\,d\tau \tag{18}$$

We may point out that in going from Eqs. (17) to (18), we have replaced $a(\tau)$ by $a(t)$ since $a(t)$ is assumed to be monotonic and $v(\lambda, 0, \tau) = 0$ for $\lambda > a(\tau)$.

Now, since, $G_1(0) \neq 0$, it follows that there exists a unique function $G_1{}^{-1}(t)$, for which

$$G_1{}^{-1}(t)*dG_1(t) = G_1(t)*dG_1{}^{-1}(t) = H(t) \tag{19}$$

On combining Eqs. (17) and (18) and employing this result in the equation we obtain

$$\int_\rho^{a(t)} v(\lambda, 0, t)\,d\lambda = \frac{4}{\pi}\int_0^t G_1{}^{-1}(t-\tau)\frac{\partial}{\partial\tau}\int_\rho^{a(t)} \frac{(\eta^2-\rho^2)^{1/2}}{\eta^2}\,d\eta \times \int_0^\eta \frac{m^2 S(m, t)}{(\eta^2-m^2)^{1/2}}\,dm\,d\tau \tag{20}$$

Finally, on differentiating Eq. (20) with respect to ρ and further simplifying, we arrive at

$$v(\rho, 0, t) = u_\theta(\rho, 0, t) = -\frac{4}{\pi}\int_0^t G_1{}^{-1}(t-\tau)\frac{\partial}{\partial\tau} \times \left[\frac{\partial}{\partial\rho}\int_\rho^{a(t)} \frac{(\eta^2-\rho^2)^{1/2}}{\eta^2}\,d\eta \int_0^\eta \frac{m^2 S(m, t)}{(\eta^2-m^2)^{1/2}}\,dm\right]d\tau \qquad \rho \leqq a(t) \tag{21}$$

Equation (21) gives the displacement over the surface of the crack in the linear viscoelastic medium for a general loading $S(\rho, t)$. The corresponding stress component $\sigma_{\theta z}(\rho, 0, t)$ can be evaluated by employing Eq. (21) in Eq. (11). If, in particular, $S(\rho, t)$ is independent of ρ, i.e., $S(\rho, t) = S(t)$ then from Eq. (21) we find

$$u_\theta(\rho, 0, t) = S(t)\left[\rho \log\left\{\frac{a(t)+[a^2(t)-\rho^2]^{1/2}}{\rho}\right\} G_1{}^{-1}(0)\right] + \int_0^t S(\tau)\rho \log\left\{\frac{a(t-\tau)+[a^2(t-\tau)-\rho^2]^{1/2}}{\rho}\right\} \frac{\partial G_1{}^{-1}(\tau)}{\partial \tau} d\tau \qquad \rho \leqq a(t) \quad (22)$$

On substituting Eq. (21) in Eq. (11) and interchanging the order of integration we obtain

$$\sigma_{\theta z}(\rho, 0, t) = -(1/2)\int_0^t G_1(t-\tau)\frac{\partial}{\partial \tau}\int_0^\tau G_1{}^{-1}(\tau-\rho)S(\tau)\frac{\partial}{\partial \rho} \times \int_0^\infty \xi^2 J_1(\xi\rho)\, d\xi \int_0^{a(\rho)} d\eta \int_0^\eta \frac{\lambda^2 J_1(\xi\lambda)}{(\eta^2-\lambda^2)^{1/2}} d\lambda\, d\rho\, d\tau \quad (23)$$

On using Eqs. (14) and (19) in the above equation and further simplifying, we finally obtain

$$\sigma_{\theta z}(\rho, 0, t) = -S(t)\left[\int_0^{a(t)} \frac{\eta H(\rho-\eta)}{\rho(\rho^2-\eta^2)^{1/2}} d\eta - (1/2)\frac{a^2(t)H[\rho-a(t)]}{\rho[\rho^2-a^2(t)]^{1/2}}\right] \quad (24)$$

For $\rho \leqq a(t)$, Eq. (24) reduces to $\sigma_{\theta z}(\rho, 0, t) = -S(t)$ as should be expected. For $\rho > a(t)$, it can be further simplified to give

$$\sigma_{\theta z}(\rho, 0, t) = \frac{S(t)}{\rho}[\rho^2-a^2(t)]^{1/2} - S(t) + \frac{S(t)a^2(t)}{2\rho[\rho^2-a^2(t)]^{1/2}} \qquad \rho > a(t) \quad (25)$$

On returning to the original problem for which the boundary condition in Eq. (4) is to be satisfied, we find

$$\sigma_{\theta z}(\rho, 0, t) = \frac{S(t)}{\rho}[\rho^2-a^2(t)]^{1/2} + \frac{S(t)}{2\rho}\frac{a^2(t)}{[\rho^2-a^2(t)]^{1/2}} \qquad \rho > a(t)$$
$$= 0 \qquad \rho \leqq a(t) \quad (26)$$

Equation (22) gives the displacement over the surface of the crack for the loading, $S(t)$. It reduces to the expression of the classical elasticity theory,[6] when the usual limits of passing from viscoelasticity to classical elasticity are employed. Equation (26) is the expression for the stress component $\sigma_{\theta z}(\rho, 0, t)$ in the plane of the crack, and a comparison of it with the one obtained in classical elasticity theory[6] indicates that there is no effect of viscoelasticity in this quantity.

The stress intensity factor $N(t)$, defined by the equation

$$N(t) = \lim_{\rho \to a^+(t)} \{2[\rho - a(t)]\}^{1/2}\sigma_{\theta z}(\rho, 0, t) \quad (27)$$

turns out to be

$$N(t) = S(t)/2\{a(t)\}^{1/2} \quad (28)$$

Also, if we define the work done in opening the crack by

$$W(t) = 2\pi \int_0^{a(t)} \rho S(t) u_\theta(\rho, 0, t)\, d\rho \quad (29)$$

by the loading $S(t)$, it follows from Eq. (22) and Eq. (29) that

$$W(t) = \frac{\pi^2 S^2(t)a^3(t)}{6G_1(0)} + \frac{\pi^2}{6}\int_0^t S^2(\tau)a^3(t-\tau)\frac{\partial}{\partial \tau}G_1{}^{-1}(\tau)\, d\tau \quad (30)$$

In the usual limits, if we identify $G_1(0)$ by 2μ, we note that first expression on the right-hand side of Eq. (30) is the same as in the classical elasticity theory.[6] The second expression on the right-hand side of Eq. (30) therefore represents the excess amount of work required due to the energy dissipation in the viscoelastic medium.

In order to have some quantitative feeling about the above results we now assume that the viscoelastic medium is a Maxwell material whose behavior is characterized by

$$G_1(t) = G_0 e^{-t/\tau_0} \qquad G_1{}^{-1}(t) = \left(\frac{1}{G_0}\right)\left(1 + \frac{t}{\tau_0}\right) \quad (31)$$

Hence, it follows that

$$G_1{}^{-1}(0) = \frac{1}{G_0} \qquad \frac{\partial}{\partial \tau}G_1{}^{-1}(\tau) = (1/G_0\tau_0) \quad (32)$$

On employing Eq. (32) in Eq. (22), we obtain

$$u_\theta(\rho, 0, t) = \frac{S(t)}{G_0}\rho \log\left[\frac{a(t)+\{a^2(t)-\rho^2\}^{1/2}}{\rho}\right] + \frac{1}{G_0\tau_0}\int_0^t S(\tau)\rho \log\left[\frac{a(\theta)+\{a^2(\theta)-\rho^2\}^{1/2}}{\rho}\right] d\theta \qquad \rho \leqq a(t) \quad (33)$$

where a change of variable has been made in writing Eq. (33). Similarly, Eq. (30) reduces to

$$W(t) = \frac{\pi^2 S^2(t)a^3(t)}{6G_0} + \frac{\pi^2}{6G_0\tau_0}\int_0^t S^2(\theta)a^3(\theta)\, d\theta \quad (34)$$

References

[1] Graham, G. A. C., "The Correspondence Principle of Linear Viscoelasticity Theory for Mixed Boundary Value Problems Involving Time-Dependent Boundary Regions," *Quarterly of Applied Mathematics*, Vol. 26, No. 2, July 1968, pp. 167–174.

[2] Graham, G. A. C., "Two Extending Coach Problems in Linear Viscoelasticity Theory," *Quarterly of Applied Mathematics*, Vol. 27, No. 4, Jan. 1970, pp. 497–507.

[3] Ting, T. C. T., "The Contact Stresses Between a Rigid Indenter and a Viscoelastic Half Space," *Journal of Applied Mechanics*, Vol. 33, No. 4, Dec. 1966, pp. 845–854.

[4] Kaloni, P. N. and Smith, R., "Penny Shaped Crack in a Linear Viscoelastic Medium Under Pure Shear," *International Journal of Solids and Structures*, Vol. 10, No. 10, Oct. 1974, pp. 1125–1137.

[5] Watson, G. N., *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, England, 1958.

[6] Sneddon, I. N. and Lowengrub, M., *Crack Problems in the Classical Theory of Elasticity*, Wiley, New York, 1969.

Surface Heat Transfer in the Flow of Dissociated Nitrogen

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THE possibility of determining the freestream atom concentrations by heat-transfer measurements at surfaces of known catalycities was investigated by a number of authors. A review of these studies was given by Chung.[1] The known calculations determining the effect of variable wall catalycities on heat transfer are based on the frozen flow assumption which simplifies the mathematics of the problem appreciably. A double surface gage with a noncatalytic and a highly catalytic surface located one after each other on a flat plate was numerically investigated by Hayday and McGraw,[2] where variable transport properties were taken into account.

The accuracy of such a diagnosis depends much on the possibility of realizing a flow configuration, where the gas phase reactions are essentially frozen, while the surface reactions are very fast. Nevertheless, in the boundary-layer flow over a highly cooled flat plate, gas phase recombination reactions are expected to occur even close to the leading edge.

For this reason numerical calculations including variable transport properties and nonequilibrium gas phase reactions were carried out for the flow of dissociated pure nitrogen over a

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